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On the Description of Quantum Vortices in Superfluid Films

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Abstract

The dynamics of vortices in a superfluid film at the absolute zero of temperature is studied. The quantum-mechanical, i.e., “first-quantized” description is compared to a recently proposed quantum-field-theoretic, i.e., “second-quantized” description. The theory, consisting of the nonrelativistic effective action of phonons linearly coupled to a Chern-Simons term, is used to calculate the one-loop amplitude for the elastic scattering of phonons from a vortex.

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1 Introduction

Vortices play an important role in several two-dimensional condensed matter systems at low temperatures such as superfluids and superconductors, fractional quantized Hall systems [1] and Josephson junction arrays [2]. They are also believed to play an important role in the quantum phase transitions these systems undergo [3, 4, 5]. Two-dimensional vortices can ideally be represented as pointlike and are, close to the absolute zero of temperature, governed by the laws of quantum physics. The quantum mechanics of such pointlike vortices, which is known since long (see, for example, Ref. [6]), is unusual. This is because classically, instead of obeying Newton's equations of motion, the coordinates of a vortex center obey Kirchoff's equations of motion [7]. As will be recalled in Sec. 3, this is connected to a peculiar canonical structure known to arise also in the problem of a charged particle confined to move in a plane subject to an external magnetic field [8]. As a result, the classical dynamics of a vortex in a two-dimensional fluid resembles that of a charged particle in a plane under the influence of a magnetic field [7].

In recent years, various quantum-mechanical formulations of vortex dynamics in superfluid films have been put forward which use this analogy [9, 10, 11]. In these so-called dual approaches (see Sec. 3), vortices are still described by delta functions representing their worldlines, but their interaction is now pictured as being mediated by a photon—the quantum of the electromagnetic field—and not, as in the original formulation, by a phonon—the quantum of the Goldstone field associated with the spontaneously broken U(1) symmetry of a superfluid. The analogy can be understood [12] by comparing the continuity equation

$$\partial_0 n + \nabla \cdot \mathbf{j} = 0, \quad (1)$$

where n is the particle number density and \mathbf{j} the particle number current of the superfluid, to the Maxwell equation (Gaussian units will be used throughout):

$$\frac{1}{c} \partial_0 H + \nabla \times \mathbf{E} = 0. \quad (2)$$

It is to be noted that in 2+1 dimensions, the magnetic field H is a scalar, and $\nabla \times \mathbf{E} = \epsilon_{ij} \partial_i E_j$, with ϵ_{ij} ($i, j = 1, 2$) the antisymmetric Levi-Civita symbol ($\epsilon_{12} = 1$). It follows that with the replacements

$$n \rightarrow H/\Phi_0, \quad j_i \rightarrow \frac{e}{h} \epsilon_{ij} E_j, \quad (3)$$

where $\Phi_0 = hc/e$ is the magnetic flux quantum, the continuity equation goes over into the Maxwell equation. The fundamental constants are included in (3) to match the dimensions on both sides of the replacements. Since the integral $\Phi(S) = \int_S d^2x H$ represents the magnetic flux penetrating the surface S , the ratio H/Φ_0 is the flux number density n_\otimes . In other words, the duality transformation maps the particles of the original formulation onto flux quanta in the dual formulation.

In this paper, we take an entirely different approach. Instead of considering a quantum-mechanical, i.e., “first-quantized” description of vortices in a superfluid film

at the absolute zero of temperature, we study a quantum-field-theoretic, i.e., “second-quantized” description recently put forward by these authors [13]. Vortices are here not described by delta functions representing their worldlines as in the quantum-mechanical approach, but by a nonsingular quantum field. The quantum field theory fulfills two stringent constraints. The first one arises because at zero temperature the entire system is superfluid, and can, therefore, be considered an ideal fluid. According to a theorem due to Helmholtz [7], a vortex in an ideal fluid moves with the fluid, and, hence, has no independent dynamics. The second constraint stems from the following observation. An electron traversing a closed path in a magnetic field accumulates an Aharonov-Bohm phase given by $(2\pi \text{ times})$ the number of flux quanta encircled [14]. The above analogy then asserts that an elementary vortex traversing a closed path in a superfluid film also accumulates a geometrical phase given by $(2\pi \text{ times})$ the number of particles encircled [15]. It has been shown in [13] (see also Sec. 4) that both these constraints are satisfied by a specific Chern-Simons theory. Two of the main characteristics of a Chern-Simons theory in general are that, first, it contains a vector field which has no independent dynamics, and that, second, it imparts flux to particles. By representing vortices by a Chern-Simons field, we assured that they have no independent dynamics, and that they see the particles as sources of geometrical phase. To connect with the standard quantum-mechanical approach, we introduce in Sec. 4 external vortices in the field theory and show that, in this way, the known results are reproduced.

The specific Chern-Simons theory proposed in [13] is based on the effective theory of phonons describing a superfluid film at low energies and small momenta (see Sec. 2). The effective Lagrangian is the most general one consistent with the symmetries of the problem, in particular Galilei invariance and invariance under global $U(1)$ transformations. In Sec. 5, we use this effective theory extended by the Chern-Simons part, representing the vortices, to investigate one-loop contributions to the elastic scattering of phonons by a vortex. We apply dimensional analysis to identify the relevant Feynman graphs. To the order in which we are working, we do not encounter ultraviolet divergences in the effective theory; only a logarithmic infrared divergence arises. This introduces an arbitrary scale factor into the problem. Often such a factor appears instead when eliminating ultraviolet divergences by a redefinition of certain parameters of the effective theory. The one-loop contributions evaluated in this paper give the first correction to an old result for the scattering of a sound wave from a vortex obtained by Pitaevskii in the Born approximation [16].

Notation In the main text we will keep track of the fundamental constants. We will follow standard notation and use \mathbf{p} as momentum variable and \mathbf{k} as wave-vector variable; they are related via $\mathbf{p} = \hbar\mathbf{k}$, where $\hbar = h/2\pi$ with h Planck’s constant. The (angular) frequency will be denoted by ω and the corresponding energy by p_0 ; $p_0 = \hbar\omega$. A spacetime point will be indicated as $x = x_\mu = (t, \mathbf{x})$, $\mu = 0, 1, \dots, d$, with d the number of space dimensions, the energy-momentum as $p = p_\mu = (p_0, \mathbf{p})$, and $k = k_\mu = (\omega, \mathbf{k})$.

2 Effective Theory for Sound Waves

In this section, we briefly recall the nonrelativistic effective action describing sound waves at low energy and small momentum in two space dimensions [17]. As starting point to describe the system, we take the microscopic model [18]

$$\mathcal{L} = \phi^* [i\hbar\partial_0 - \epsilon(-i\hbar\nabla) + \mu_0] \phi - \lambda_0 |\phi|^4, \quad (4)$$

where the complex scalar field $\phi(x)$ describes the atoms of mass m constituting the liquid, $\epsilon(-i\hbar\nabla) = -\hbar^2\nabla^2/2m$ is the kinetic energy operator, and μ_0 is the chemical potential. The last term with a positive coupling constant, $\lambda_0 > 0$, represents a repulsive contact interaction.

The Lagrangian has a global U(1) symmetry, under which the matter field transforms as

$$\phi(x) \rightarrow \phi'(x) = e^{i\alpha} \phi(x), \quad (5)$$

with α the transformation parameter. At zero temperature, this global symmetry is spontaneously broken by a nontrivial ground state. This can be easily seen by considering the potential energy

$$\mathcal{V} = -\mu_0 |\phi|^2 + \lambda_0 |\phi|^4, \quad (6)$$

which has, for $\mu_0 > 0$, a minimum away from the origin $\phi = 0$ at

$$|\bar{\phi}|^2 = \frac{1}{2} \frac{\mu_0}{\lambda_0}. \quad (7)$$

Since the total particle number density $n(x)$ is given by

$$n(x) = |\phi(x)|^2, \quad (8)$$

the quantity $\bar{n}_0 := |\bar{\phi}|^2$ physically represents the number density of particles contained in the condensate.

To account for the nontrivial ground state, we shift ϕ by the (complex) constant $\bar{\phi}$ and write [9]

$$\phi(x) = e^{i\varphi(x)} [\bar{\phi} + \tilde{\phi}(x)]. \quad (9)$$

The scalar field $\varphi(x)$ is a background field representing the Goldstone mode of the spontaneously broken global U(1) symmetry. In terms of the new variables, the quadratic terms of the Lagrangian (4) may be cast in the matrix form

$$\mathcal{L}_0 = \frac{1}{2} \tilde{\Phi}^\dagger M_0 \tilde{\Phi}, \quad \tilde{\Phi} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi}^* \end{pmatrix}, \quad (10)$$

with

$$M_0 = \begin{pmatrix} i\hbar\partial_0 - \epsilon + \mu_0 - U - 4\lambda_0 |\bar{\phi}|^2 & -2\lambda_0 \bar{\phi}^2 \\ -2\lambda_0 \tilde{\phi}^{*2} & -i\hbar\partial_0 - \epsilon + \mu_0 - U - 4\lambda_0 |\bar{\phi}|^2 \end{pmatrix}, \quad (11)$$

and U is the Galilei-invariant combination [9]

$$U = \hbar\partial_0\varphi + \frac{1}{2m}(\hbar\nabla\varphi)^2. \quad (12)$$

In writing (11), we have omitted a term of the form $\nabla^2\varphi$ containing two derivatives which is irrelevant in the regime of low momentum in which we are interested. We have also omitted a term of the form $\nabla\varphi \cdot \mathbf{j}$, where \mathbf{j} is the Noether current associated with the global U(1) symmetry,

$$\mathbf{j} = \frac{\hbar}{2im}(\phi^*\nabla\phi - \nabla\phi^*\phi). \quad (13)$$

This term, which after a partial integration becomes $-\varphi\nabla \cdot \mathbf{j}$, is also irrelevant at low energy and small momentum because in a first approximation the particle number density is constant, so that the classical current satisfies the condition

$$\nabla \cdot \mathbf{j} = 0. \quad (14)$$

The spectrum $E(\mathbf{p})$ obtained from the matrix M_0 with the background field U set to zero is the famous single-particle Bogoliubov spectrum [19],

$$\begin{aligned} E(\mathbf{p}) &= \sqrt{\epsilon^2(\mathbf{p}) + 2\mu_0\epsilon(\mathbf{p})} \\ &= \sqrt{\epsilon^2(\mathbf{p}) + 4\lambda_0|\bar{\phi}|^2\epsilon(\mathbf{p})}. \end{aligned} \quad (15)$$

The most notable feature of this spectrum is that it is gapless, behaving for small momentum as

$$E(\mathbf{p}) \sim u_0 |\mathbf{p}|, \quad (16)$$

with $u_0 = \sqrt{\mu_0/m}$ a velocity which is sometimes referred to as the microscopic sound velocity. It was first shown by Beliaev [18] that (in three dimensions) the gaplessness of the single-particle spectrum persists at the one-loop order. This was subsequently proven to hold to all orders in perturbation theory by Hugenholtz and Pines [20]. For large momentum, the Bogoliubov spectrum takes a form

$$E(\mathbf{p}) \sim \epsilon(\mathbf{p}) + 2\lambda_0|\bar{\phi}|^2 \quad (17)$$

typical for a nonrelativistic particle with mass m moving in a medium. To highlight the condensate, we have chosen here the second equality in (15), where μ_0 is replaced with $2\lambda_0|\bar{\phi}|^2$.

The gaplessness of the single-particle spectrum (15) is a result of Goldstone's theorem. This can, for example, be seen by considering the relativistic version of the theory. There, one finds two spectra: one corresponding to a massive Higgs particle which in the nonrelativistic limit becomes extremely heavy and decouples from the theory, and one corresponding to the Goldstone mode of the spontaneously broken global U(1) symmetry [21]. The latter reduces in the nonrelativistic limit to the Bogoliubov spectrum. Another way to see this is to couple the theory (4) to an electromagnetic field. The single-particle spectrum then acquires an energy gap. This is what one expects to happen to the spectrum of a Goldstone mode when the Higgs mechanism is operating. In other words, the single-particle spectrum (15) is identical to that of the Goldstone mode which physically represents the collective density fluctuations in the superfluid. The equivalence of the single-particle excitation and the collective density fluctuation has been proven to all orders in perturbation by Gavoret and Nozières [22].



Figure 1: Graphical representation of the effective theory (24). The symbols are explained in the text.

To find the effective theory governing the superfluid at low energy and small momentum from (10), we integrate out the fluctuating field $\tilde{\Phi}$ [17]. The first two terms of the effective theory are graphically represented by Fig. 1. A line with a shaded bubble inserted stands for $i\hbar$ times the *full* Green function G and the black bubble denotes i/\hbar times the *full* interaction Γ of the $\tilde{\Phi}$ -field with the background field U which is denoted by a wiggly line. Both G and Γ are 2×2 matrices. The full interaction is obtained from the inverse Green function by differentiation with respect to the chemical potential,

$$\Gamma = -\frac{\partial G^{-1}}{\partial \mu}. \quad (18)$$

This follows because U , as defined in (12), appears in the theory only in the combination $\mu - U$. To lowest order, the inverse Green function is given by the matrix M_0 in (11), so that the vertex describing the interaction between the $\tilde{\Phi}$ and U -fields is i/\hbar times minus the unit matrix. Because in terms of the full Green function G , the particle number density reads

$$\bar{n} = \frac{i\hbar}{2} \text{tr} \int \frac{d^{d+1}p}{h^{d+1}} G(p), \quad (19)$$

where $d^{d+1} = dp_0 d^d p$, it follows that the first diagram in Fig. 1 stands for $-\bar{n}U$. The bar over n is to show that the particle number density obtained in this way is a constant, representing the density of the uniform system with $U(x)$ set to zero. The second diagram without the wiggly lines denotes i/\hbar times the (0 0)-component of the *full* polarization tensor, Π_{00} , at zero energy transfer and low momentum \mathbf{q} ,

$$\frac{i}{\hbar} \lim_{\mathbf{q} \rightarrow 0} \Pi_{00}(0, \mathbf{q}) = -\frac{1}{2} \lim_{\mathbf{q} \rightarrow 0} \text{tr} \int \frac{d^{d+1}p}{h^{d+1}} G \Gamma G(p_0, \mathbf{p} + \mathbf{q}). \quad (20)$$

The factor $\frac{1}{2}$ is a symmetry factor which arises because the two Bose lines are identical. We proceed by invoking an argument due to Gavoret and Nozières [22] to relate the left-hand side of (20) to the sound velocity. By virtue of relation (18) between the full Green function G and the full interaction Γ , the (0 0)-component of the polarization tensor can be cast in the form

$$\begin{aligned} \lim_{\mathbf{q} \rightarrow 0} \Pi_{00}(0, \mathbf{q}) &= -\frac{i\hbar}{2} \lim_{\mathbf{q} \rightarrow 0} \text{tr} \int \frac{d^{d+1}p}{h^{d+1}} G \frac{\partial G^{-1}}{\partial \mu} G(p_0, \mathbf{p} + \mathbf{q}) \\ &= \frac{i\hbar}{2} \frac{\partial}{\partial \mu} \lim_{\mathbf{q} \rightarrow 0} \text{tr} \int \frac{d^{d+1}p}{h^{d+1}} G(p_0, \mathbf{p} + \mathbf{q}) \end{aligned}$$

$$= \frac{\partial \bar{n}}{\partial \mu} = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu^2}, \quad (21)$$

where Ω is the thermodynamic potential and V the volume of the system. The right-hand side of (21) is $\bar{n}^2 \kappa$, with κ the compressibility. Because it is related to the macroscopic sound velocity c via

$$\kappa = \frac{1}{m \bar{n} c^2}, \quad (22)$$

we conclude that the (0 0)-component of the full polarization tensor satisfies the so-called compressibility sum rule of statistical physics [22]

$$\lim_{\mathbf{q} \rightarrow 0} \Pi_{00}(0, \mathbf{q}) = \frac{\bar{n}}{m c^2}. \quad (23)$$

Putting the pieces together, we infer that the diagrams in Fig. 1 stand for the effective theory [17]

$$\mathcal{L}_{\text{eff}} = -\bar{n} \left[\hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right] + \frac{\bar{n}}{2m c^2} \left[\hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right]^2, \quad (24)$$

where we recall that \bar{n} is the particle number density of the fluid at rest. This nonlinear theory describes a nonrelativistic sound wave, with the dimensionless real scalar field φ representing the Goldstone mode of the spontaneously broken global U(1) symmetry. It has the gapless dispersion relation $\omega^2(\mathbf{k}) = c^2 \mathbf{k}^2$, where ω is the (angular) frequency, \mathbf{k} the wave vector, and c the sound velocity. This effective theory gives a complete description of the superfluid valid at low energies and small momenta. There are of course higher-order terms, but they need to be included only as one goes to higher energies and momenta. The same effective theory we discussed here appears also in the context of (neutral) superconductors and that of classical hydrodynamics [23].

The form of the effective theory (24) can also be derived from general symmetry arguments [24, 25]. The basic idea is that the presence of a gapless Goldstone has to be reconciled with Galilei invariance, which demands that the mass current and the momentum density are equal. This leads to the conclusion that the U(1) Goldstone field φ can only appear in the combination (12). To obtain the required linear spectrum for the Goldstone mode it is necessary then to have the form (24).

The particle number current that follows from (24) reads

$$n(x) = \bar{n} - \frac{\bar{n}}{m c^2} \left\{ \hbar \partial_0 \varphi(x) + \frac{1}{2m} [\hbar \nabla \varphi(x)]^2 \right\} \quad (25)$$

$$\mathbf{j}(x) = n(x) \mathbf{v}(x), \quad (26)$$

where $\mathbf{v} = (\hbar/m) \nabla \varphi$ is the superfluid velocity field. Equation (25) reflects Bernoulli's principle which states that in regions of rapid flow, the density and therefore the pressure is low.

The diagrams of Fig. 1 can also be evaluated in a loop expansion to obtain explicit expressions for the particle number density \bar{n} and the sound velocity c to a given order [17]. In doing so, one encounters—apart from ultraviolet divergences which can be eliminated by renormalization—also infrared divergences because the Bogoliubov

spectrum is gapless. When, however, all one-loop contributions are added together, these divergences are seen to cancel [17]. One finds for $d = 2$ to the one-loop order

$$\bar{n} = \frac{1}{2} \frac{\mu}{\lambda}, \quad c^2 = \frac{\mu}{m} = 2 \frac{\lambda}{m} \bar{n}, \quad (27)$$

where λ and μ are the renormalized coupling constants (see below). The second expression in (27) for the sound velocity is appropriate when, as is often the case in experiment, the particle number is fixed. The first one, featuring the chemical potential, is appropriate in the presence of a reservoir with which the system can freely exchange particles, so that only the *average* particle number is fixed.

Given the form of the effective theory, the particle number density and sound velocity can be more easily obtained directly from the thermodynamic potential Ω via

$$\bar{n} = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu}; \quad \frac{1}{c^2} = -\frac{1}{V} \frac{m}{\bar{n}} \frac{\partial^2 \Omega}{\partial \mu^2}, \quad (28)$$

where V is the volume of the system. In this approach, one only has to calculate the thermodynamic potential which at zero temperature and in the approximation we are working is given by the effective potential \mathcal{V}_{eff} corresponding to the theory (4): $\Omega = \int d^d x \mathcal{V}_{\text{eff}}$. To the one-loop order, the effective potential for a uniform system reads

$$\mathcal{V}_{\text{eff}} = -\frac{\mu_0^2}{4\lambda_0} + \frac{1}{2} \int \frac{d^d p}{h^d} E(\mathbf{p}), \quad (29)$$

with $E(\mathbf{p})$ the gapless Bogoliubov spectrum (15). The integral over the loop momentum in arbitrary space dimension d yields

$$\mathcal{V}_{\text{eff}} = -\frac{\mu_0^2}{4\lambda_0} - \frac{\Gamma(1-d/2)\Gamma(d/2+1/2)}{2\pi^{(d+1)/2}\hbar^d \Gamma(d/2+2)} m^{d/2} \mu_0^{d/2+1}, \quad (30)$$

where we used the integral representation of the Gamma function

$$\frac{1}{a^z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{d\tau}{\tau} \tau^z e^{-a\tau} \quad (31)$$

together with dimensional regularization to suppress irrelevant ultraviolet divergences. By this we mean contributions which, for regularization with a momentum cutoff Λ , would diverge with a strictly positive power of Λ . Note that we, following Ref. [26], first carried out the integrals over the loop energies, and then analytically continued the remaining integrals over the loop momenta to arbitrary space dimensions d .

Expanded around $d = 2$, Eq. (30) gives

$$\mathcal{V}_{\text{eff}} = -\frac{\mu_0^2}{4\lambda_0} - \frac{1}{4\pi\hbar^2\epsilon} \frac{m\mu_0^2}{\kappa^\epsilon} + \mathcal{O}(\epsilon^0), \quad (32)$$

where $\epsilon = 2 - d$ is the deviation from the upper critical dimension $d = 2$, and κ is an arbitrary renormalization group scale parameter, with the dimension of an inverse

length. The right-hand side of (32) is seen to diverge in the limit $d \rightarrow 2$. The theory can be rendered ultraviolet finite by introducing a renormalized coupling constant λ via

$$\frac{1}{\lambda_0} = \frac{1}{\kappa^\epsilon} \left(\frac{1}{\hat{\lambda}} - \frac{m}{\pi \hbar^2 \epsilon} \right), \quad (33)$$

where $\hat{\lambda} = \lambda/\kappa^\epsilon$. Its definition is such that for arbitrary d , $\hat{\lambda}$ has the same engineering dimension as λ_0 in the upper critical dimension $d = 2$. As renormalization prescription we used the modified minimal subtraction. The beta function $\beta(\hat{\lambda})$ follows as [27]

$$\beta(\hat{\lambda}) = \kappa \left. \frac{\partial \hat{\lambda}}{\partial \kappa} \right|_{\lambda_0} = -\epsilon \hat{\lambda} + \frac{m}{\pi \hbar^2} \hat{\lambda}^2. \quad (34)$$

In the upper critical dimension, this yields only one fixed point, viz. the infrared-stable fixed point $\hat{\lambda}^* = 0$. Below $d = 2$, this point is shifted to $\hat{\lambda}^* = \epsilon \pi \hbar^2 / m$.

It follows from (32) that the chemical potential is not renormalized to this order. Incidentally, from the view point of renormalization, the mass m is an irrelevant parameter in nonrelativistic theories and can be scaled away.

The most remarkable aspect of the effective theory (4) is that it is nonlinear. The nonlinearity is necessary to provide a Galilei-invariant description of a gapless mode in a nonrelativistic system. Since the Goldstone field in (4) is always accompanied by a derivative, we see that the nonlinear terms carry additional factors of $\hbar|\mathbf{k}|/mc$, with $|\mathbf{k}|$ the wave number. They can therefore be ignored provided the wave number is smaller than the inverse coherence length $\xi = \hbar/mc$,

$$|\mathbf{k}| < 1/\xi. \quad (35)$$

For example, for ^4He the coherence length, or Compton wavelength, is about 10 nm. In this system, the bound (35), below which the nonlinear terms can be neglected, coincide with the region where the spectrum is linear and the description in terms of a sound mode is applicable.

We close this section discussing the apparent mismatch in the number of degrees of freedom in the normal and superfluid state. Whereas the normal state is described by a complex field ϕ , the superfluid state is described by just a single real scalar field φ . The resolution of this paradox lies in the spectrum of the modes [28]. In the normal state, the spectrum $E(\mathbf{p}) = \mathbf{p}^2/2m$ is linear in E , so that only positive energies appear in the Fourier decomposition of the field ϕ . One needs therefore a complex field to describe a single particle, as is well known from standard quantum mechanics. In the superfluid state, where the spectrum $E^2(\mathbf{p}) = c^2 \mathbf{p}^2$, is quadratic in E , the Fourier decomposition of the field φ contains positive as well as negative energies. As a result, a single real field suffices to describe this mode. In other words, although the number of fields is different, the number of degrees of freedom is the same in the normal and superfluid state.

3 Quantum Mechanics of Vortices

In this section, we discuss the conventional quantum-mechanical, or “first-quantized” description of vortices. We will be working at the absolute zero of temperature, so that

the entire liquid is superfluid. In the microscopic theory (4), the asymptotic solution of a static vortex with winding number w centered at the origin is well known [29]

$$\phi(\mathbf{x}) = \sqrt{\frac{\mu}{2\lambda}} \left(1 - \xi^2 \frac{w^2}{4\mathbf{x}^2} \right) e^{iw\theta} + \mathcal{O}\left(\frac{1}{\mathbf{x}^4}\right), \quad (36)$$

where θ is the azimuthal angle and ξ is the coherence length introduced above (35) which because of (27) can also be written as $\xi = \hbar/\sqrt{m\mu}$. The density profile $n(\mathbf{x})$ in the presence of this vortex follows from taking $|\phi(\mathbf{x})|^2$.

Let us now discuss how vortices can be incorporated in the effective theory. To this end we follow Kleinert [30] and introduce a so-called plastic field $\varphi_\mu^P = (\varphi_0^P, \varphi^P)$ in the effective theory (24) via minimally coupling to the Goldstone field:

$$\tilde{\partial}_\mu \varphi \rightarrow \tilde{\partial}_\mu \varphi + \varphi_\mu^P, \quad (37)$$

with $\tilde{\partial}_\mu = (\partial_0, -\nabla)$. If there are N vortices with winding number w_α ($\alpha = 1, \dots, N$) and centered at $\mathbf{X}^1(t), \dots, \mathbf{X}^N(t)$, the plastic field satisfies the relation

$$\nabla \times \varphi^P(x) = -2\pi \sum_\alpha w_\alpha \delta[\mathbf{x} - \mathbf{X}^\alpha(t)], \quad (38)$$

so that we obtain for the velocity field

$$\nabla \times \mathbf{v} = \sum_\alpha \gamma_\alpha \delta[\mathbf{x} - \mathbf{X}^\alpha(t)], \quad (39)$$

as required. Here, $\gamma_\alpha = (h/m)w_\alpha$, with $w_\alpha = 0, \pm 1, \pm 2, \dots$, denotes the circulation of the α th vortex which is quantized in units of h/m . A summation over vortex labels will always be made explicit. The combination $\tilde{\partial}_\mu \varphi + \varphi_\mu^P$ is invariant under the local gauge transformation

$$\varphi(x) \rightarrow \varphi(x) + \alpha(x); \quad \varphi_\mu^P \rightarrow \varphi_\mu^P - \tilde{\partial}_\mu \alpha(x), \quad (40)$$

with φ_μ^P playing the role of a gauge field.

In the gauge $\varphi_0^P = 0$, a solution of Eq. (38) is given by

$$\varphi_i^P(x) = 2\pi \epsilon_{ij} \sum_\alpha w_\alpha \delta_j[x, L_\alpha(t)], \quad (41)$$

where ϵ_{ij} is the antisymmetric Levi-Civita symbol in two dimensions, with $\epsilon_{12} = 1$, and $\delta[x, L_\alpha(t)]$ is a delta function on the line $L_\alpha(t)$ starting at the position $\mathbf{X}^\alpha(t)$ of the α th vortex and running to spatial infinity along an arbitrary path:

$$\delta_i[x, L_\alpha(t)] = \int_{L_\alpha(t)} dy_i \delta(\mathbf{x} - \mathbf{y}). \quad (42)$$

Let us for the moment concentrate on static vortices. The field equation obtained from the effective theory (24) with $\nabla \varphi$ replaced by the covariant derivative $\nabla \varphi - \varphi^P$ and $\partial_0 \varphi$ set to zero simply reads

$$\nabla \cdot \mathbf{v} = 0, \quad \text{or} \quad \nabla \cdot (\nabla \varphi - \varphi^P) = 0, \quad (43)$$

when the fourth-order term is neglected. It can be easily solved to yield

$$\varphi(\mathbf{x}) = - \int d^2y G(\mathbf{x} - \mathbf{y}) \nabla \cdot \boldsymbol{\varphi}^P(\mathbf{y}), \quad (44)$$

where $G(\mathbf{x})$ is the Green function of the Laplace operator in two space dimensions

$$G(\mathbf{x}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{k^2} = -\frac{1}{2\pi} \ln(|\mathbf{x}|), \quad (45)$$

i.e., $\nabla^2 G(\mathbf{x}) = -\delta(\mathbf{x})$. For the velocity field we obtain in this way the well-known expression [7]

$$\begin{aligned} v_i(\mathbf{x}) &= \epsilon_{ij} \sum_{\alpha} \gamma_{\alpha} \partial_j \int d^2y G(\mathbf{x} - \mathbf{y}) \delta[\mathbf{y} - \mathbf{X}^{\alpha}(t)] \\ &= -\frac{1}{2\pi} \epsilon_{ij} \sum_{\alpha} \gamma_{\alpha} \frac{x_j - X_j^{\alpha}}{|\mathbf{x} - \mathbf{X}^{\alpha}|^2}, \end{aligned} \quad (46)$$

which is valid for \mathbf{x} sufficiently far away from the vortex cores. Let us now specialize to the case of a single static vortex centered at the origin. On substituting the corresponding solution in (25), we find for the density profile in the presence of a static vortex asymptotically

$$n(\mathbf{x}) = \bar{n} \left(1 - \xi^2 \frac{w^2}{2\mathbf{x}^2} \right). \quad (47)$$

Since this density profile is identical to the one obtained in the microscopic theory [see Eq. (36)], this exemplifies that, with the aid of plastic fields [30], vortices are correctly accounted for in the effective theory.

Let us proceed to investigate the dynamics of vortices in this formalism and derive the action governing it. We consider only the first part of the effective theory (24). In ignoring the higher-order terms, we approximate the superfluid by an incompressible fluid for which the particle number density is constant, $n(x) = \bar{n}$, see Eq. (25). This is an approximation often used. We again work in the gauge $\varphi_0^P = 0$ and replace $\nabla\varphi$ by the covariant derivative $\nabla\varphi - \boldsymbol{\varphi}^P$, with the plastic field given by (38). The solution of the resulting field equation for φ is again of the form (44), but now it is time-dependent because the plastic field is. Substituting this in the action $S_{\text{eff}} = \int dt d^2x \mathcal{L}_{\text{eff}}$, we find after some straightforward calculus

$$S_{\text{eff}} = m\bar{n} \int dt \left[-\frac{1}{2} \sum_{\alpha} \gamma_{\alpha} \mathbf{X}^{\alpha} \times \dot{\mathbf{X}}^{\alpha} + \frac{1}{2\pi} \sum_{\alpha < \beta} \gamma_{\alpha} \gamma_{\beta} \ln(|\mathbf{X}^{\alpha} - \mathbf{X}^{\beta}|) \right], \quad (48)$$

where a dot over a symbol denotes the time derivative. This action yields the well-known equations of motion for point vortices in an incompressible two-dimensional superfluid [7, 31]:

$$\dot{X}_i^{\beta}(t) = -\frac{1}{2\pi} \epsilon_{ij} \sum_{\alpha \neq \beta} \gamma_{\alpha} \frac{X_j^{\beta}(t) - X_j^{\alpha}(t)}{|\mathbf{X}^{\beta}(t) - \mathbf{X}^{\alpha}(t)|^2}. \quad (49)$$

Note that $\dot{X}_i^\beta(t) = v_i [\mathbf{X}^\beta(t)]$, where $\mathbf{v}(x)$ is the superfluid velocity (46) with the time-dependence of the vortex centers included. This nicely illustrates the result due to Helmholtz for ideal fluids [7] we alluded to in the Introduction, that a vortex moves with the fluid, i.e., with the local velocity produced by the other vortices in the system. Experimental support for this conclusion has been reported in Ref. [32].

The action (48) leads to a twisted canonical structure. To display it, let us rewrite the first term of the corresponding Lagrangian as

$$L_1 = -m\bar{n} \sum_{\alpha} \gamma_{\alpha} X_1^{\alpha} \dot{X}_2^{\alpha}, \quad (50)$$

where we ignored a total derivative. It follows that the canonical conjugate to the second component X_2^{α} of the center coordinate \mathbf{X}^{α} is its first component [6]

$$\frac{\partial L_1}{\partial \dot{X}_2^{\alpha}} = -m\bar{n} \gamma_{\alpha} X_1^{\alpha}. \quad (51)$$

This implies that phase space coincides with real space and it gives the commutation relation

$$[X_1^{\alpha}, X_2^{\beta}] = \frac{i}{w_{\alpha}} \ell^2 \delta^{\alpha\beta}, \quad (52)$$

where

$$\ell = 1/\sqrt{2\pi\bar{n}} \quad (53)$$

is a characteristic length. Its definition is such that $2\pi\ell^2$ is the average area occupied by a particle of the superfluid film. The commutation relation leads to an uncertainty in the location of the vortex centers

$$\Delta X_1^{\alpha} \Delta X_2^{\alpha} \geq \frac{\ell^2}{2|w_{\alpha}|}, \quad (54)$$

which is inverse proportional to the particle number density.

Elementary quantum mechanics [33] tells us that in the quasi-classical approximation to each unit cell (of area h) in phase space there corresponds one quantum state. That is, the total number of states is given by

$$\# \text{ states} = \frac{1}{h} \int dp dq, \quad (55)$$

where p and q are a pair of canonically conjugate variables, and the integral is over the entire phase space. For a two-dimensional quantum vortex with winding number w_{β} , this implies that the number of states it can be in is

$$\# \text{ states} = |w_{\beta}| \bar{n} S, \quad (56)$$

where S is the surface area of the sample. In other words, every particle in the superfluid film makes (w_{β} times) an additional state available to the vortex.

This phenomenon that phase space coincides with real space is known to also arise in the Landau problem of a charged particle confined to move in a plane subject to a

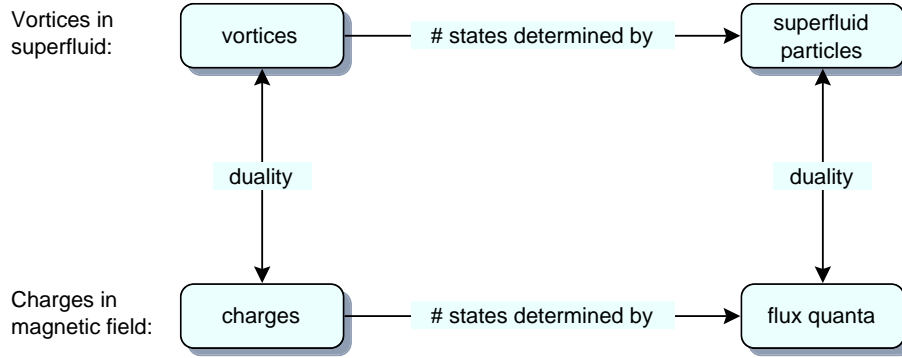


Figure 2: Duality transformation.

(constant) background magnetic field. There, it leads to the well-known result that the number of states available to the charged particle in each Landau level is

$$\# \text{ states} = |e_\beta| \frac{H}{hc} S, \quad (57)$$

where H is the magnetic field component perpendicular to the plane, and

$$e_\beta = v_\beta e_0, \quad (58)$$

($v_\beta = 0, \pm 1, \pm 2, \dots$) is the electric charge of the particle given as a multiple of the unit of charge $e_0 (> 0)$. In terms of the magnetic flux quantum $\Phi_0 = hc/e_0$, the number of states can be rewritten as

$$\# \text{ states} = |v_\beta| \frac{H}{\Phi_0} S = |v_\beta| \bar{n}_\otimes S, \quad (59)$$

where \bar{n}_\otimes is the flux number density. Hence, whereas the number of states for vortices in a superfluid film is determined by the particle number, here it is determined by the flux number. This agrees with the replacement (3) discussed in the Introduction.

As remarked there, the analogy between the two problems is the basis of a much used duality transformation. If we collectively refer to the electric charges in the Landau problem and the particles in the superfluid as ‘charges’, and to the vortices in the superfluid and the flux quanta in the Landau problem as ‘vortices’, then the duality transformation interchanges charges and vortices (see Fig. 2). Using this analogy, we see that the characteristic length (53) translates into

$$\ell_H = 1/\sqrt{2\pi\bar{n}_\otimes}, \quad (60)$$

which is precisely the magnetic length of the Landau problem.

We next turn to the geometrical phase [34]. In the two-dimensional Landau problem, when a charged particle, say the β th, is moved adiabatically around a close path

Γ_β , its wavefunction accumulates an extra Aharonov-Bohm phase factor $\gamma(\Gamma_\beta)$ given by the Wilson loop:

$$W(\Gamma_\beta) = \exp[i\gamma(\Gamma_\beta)] = \exp\left(\frac{ie_\beta}{\hbar c} \oint_{\Gamma_\beta} \mathbf{dx} \cdot \mathbf{A}\right) = \exp\left[2\pi i v_\beta \frac{HS(\Gamma_\beta)}{\Phi_0}\right], \quad (61)$$

where \mathbf{A} is the vector potential describing the background magnetic field and $HS(\Gamma_\beta)$ the magnetic flux through the area $S(\Gamma_\beta)$ spanned by the loop Γ_β . The geometrical phase $\gamma(\Gamma_\beta)$ in (61) is seen to be $(2\pi v_\beta \text{ times})$ the number of flux quanta enclosed by the path Γ_β .

Because of the above analogy, it follows that the geometrical phase acquired by the wavefunction of a vortex when it is moved adiabatically around a closed path in the superfluid film is $(2\pi w_\beta \text{ times})$ the number of particles enclosed by the path [15].

We have up to now considered a constant magnetic background field and an *incompressible* superfluid characterized by a constant particle number density. A more general analysis of the geometrical phase for a *compressible* superfluid was carried out by Haldane and Wu [15]. Their starting point was an *Ansatz* for the multivortex wavefunction of the interacting Bose condensate [35]

$$\Psi = \prod_{\alpha} A(\mathbf{X}^{\alpha}) f(\mathbf{X}^{\alpha}) \Psi^0, \quad (62)$$

where Ψ^0 describes the condensate in the absence of vortices. The operator $A(\mathbf{X})$ introduces an elementary vortex centered at \mathbf{X} ,

$$A(\mathbf{X}) = \prod_a [(x_1^a - X_1) + i(x_2^a - X_2)], \quad (63)$$

where \mathbf{x}^a denotes the coordinate of the a th particle, and the factor $f(\mathbf{X})$ accounts for the change in the particle number density owing to the presence of this vortex. Repeating steps first carried out by Arovas, Schrieffer, and Wilczek [36] in the context of the fractional quantized Hall effect, they cast the geometrical phase [34],

$$\gamma(\Gamma_\beta) = i \oint_{\Gamma_\beta} d\mathbf{X}^\beta \cdot \langle \Psi | \nabla_{\mathbf{X}^\beta} \Psi \rangle, \quad (64)$$

picked up by the wavefunction when the β th vortex is moved along a closed path Γ_β in the form

$$\gamma(\Gamma_\beta) = \int d^2x \oint_{\Gamma_\beta} d\mathbf{X}^\beta \cdot \nabla \times [\ln(|\mathbf{x} - \mathbf{X}^\beta|)] n(\mathbf{x}; \mathbf{X}^\alpha, \mathbf{X}^\beta). \quad (65)$$

Here, $n(\mathbf{x}; \mathbf{X}^\alpha, \mathbf{X}^\beta)$ is the particle number density in the presence of the transported vortex as well as the other vortices centered at \mathbf{X}^α ($\alpha \neq \beta$). Writing

$$n(\mathbf{x}; \mathbf{X}^\alpha, \mathbf{X}^\beta) = n(\mathbf{x}; \mathbf{X}^\alpha) + \delta n(\mathbf{x} - \mathbf{X}^\beta), \quad (66)$$

where $n(\mathbf{x}; \mathbf{X}^\alpha)$ is the particle number density in the absence of the vortex being moved, they concluded that, apart from corrections owing to residual vortex interactions that become small in the dilute-vortex limit, the geometrical phase is given by

$$\gamma(\Gamma_\beta) = 2\pi \int_{\Gamma_\beta} d^2x n(\mathbf{x}; \mathbf{X}^\alpha). \quad (67)$$

That is, for a compressible superfluid a similar result is found as in the simpler case of an incompressible superfluid, in that the geometrical phase acquired by an elementary vortex being moved around a closed path is (2π times) the *average* number of particles enclosed by the path [15].

The last topic we wish to discuss in this section on the quantum mechanics of vortices is the elastic scattering of a sound wave from a vortex. The subject, first studied by Pitaevskii [16], has recently received considerable attention (see Ref. [37] and references therein). To obtain the hydrodynamic equation governing this scattering process, we first record the nonlinear field equation obtained from the effective theory (24) [23]

$$\partial_0 \mu + c^2 \nabla \cdot \mathbf{v} = \partial_0 \mathbf{v}^2 + \frac{1}{2} \mathbf{v} \cdot \nabla \mathbf{v}^2, \quad (68)$$

with $\mu = -(\hbar/m)\partial_0 \varphi$ the chemical potential per unit mass. If we ignore the nonlinear terms, Eq. (68) becomes the more familiar wave equation

$$\partial_0^2 \varphi - c^2 \nabla^2 \varphi = 0. \quad (69)$$

Let us next introduce a freely moving, i.e., non-pinned vortex. Driven by the sound wave, the vortex will oscillate around some point, $\mathbf{x} = 0$ say. The velocity field in the presence of a moving vortex is obtained from the static solution [see Eq. (46)]

$$v_i(\mathbf{x}) = \frac{1}{2\pi} \gamma \epsilon_{ij} \frac{x_j}{\mathbf{x}^2}, \quad (70)$$

by replacing the coordinate \mathbf{x} with $\mathbf{x} - \mathbf{v}_L t$, where $\mathbf{v}_L(t)$ is the velocity of the vortex. This implies that

$$\partial_0 \mathbf{v}_v(\mathbf{x} - \mathbf{v}_L t) = -\mathbf{v}_L \cdot \nabla \mathbf{v}_v(\mathbf{x} - \mathbf{v}_L t), \quad (71)$$

where \mathbf{v} is given an index v to show that it is the velocity field produced by the vortex. Since the solution \mathbf{v}_v is curl-free outside the vortex core, the right-hand side of Eq. (71) may be written as $-\nabla(\mathbf{v}_L \cdot \mathbf{v}_v)$ there. The hydrodynamic equation governing the elastic scattering of a sound wave from the free vortex now follows from writing the velocity field as

$$\mathbf{v}(x) = \mathbf{v}_v(\mathbf{x} - \mathbf{v}_L t) + \frac{\hbar}{m} \nabla \tilde{\varphi}(x), \quad (72)$$

with $\tilde{\varphi}$ describing small variations around the oscillating vortex solution, and noting that (71) requires that we write for the chemical potential per unit mass

$$\mu(x) = \mathbf{v}_L \cdot \mathbf{v}_v(\mathbf{x} - \mathbf{v}_L t) - \frac{\hbar}{m} \partial_0 \tilde{\varphi}(x). \quad (73)$$

Substituting this in the field equation (68), we obtain to linear order in $\tilde{\varphi}$ the hydrodynamic equation [37],

$$\partial_0^2 \tilde{\varphi}(x) - c^2 \nabla^2 \tilde{\varphi}(x) = -\mathbf{v}_v(\mathbf{x}) \cdot \nabla \partial_0 [2\tilde{\varphi}(x) - \tilde{\varphi}(t, 0)], \quad (74)$$

where we approximated $\mathbf{v}_v(\mathbf{x} - \mathbf{v}_L t)$ by $\mathbf{v}_v(\mathbf{x})$. To linear order in $\tilde{\varphi}$ this is allowed since the vortex, being driven by the sound wave, has a velocity

$$\mathbf{v}_L(t) = \frac{\hbar}{m} \nabla \tilde{\varphi}(t, \mathbf{x} = \mathbf{v}_L t) \approx \frac{\hbar}{m} \nabla \tilde{\varphi}(t, 0). \quad (75)$$

We also neglected a term quadratic in \mathbf{v}_v which is justified because $|\mathbf{v}_v| \ll c$ outside the vortex core [16]. The first term at the right-hand side of (74) stems from the nonlinear term $\partial_0 \mathbf{v}^2$ in the field equation (68).

Equation (74) has been used by Sonin [37] as a basis to study the elastic scattering of a sound wave from a free moving vortex. In the Born approximation, he found as two-dimensional scattering amplitude $f(\theta, |\mathbf{k}|)$ the result first derived by Pitaevskii [16]:

$$f(\theta, |\mathbf{k}|) = \frac{1}{2} \sqrt{\frac{|\mathbf{k}|}{2\pi}} \frac{\hbar}{mc} e^{i\frac{\pi}{4}} \frac{\sin \theta \cos \theta}{1 - \cos \theta}, \quad (76)$$

where θ is the angle between the incoming and the scattered sound wave, and $|\mathbf{k}|$ their wave number.

In the next section, we discuss a quantum field theory of vortices recently proposed by these authors [13] and show that it reproduces the results discussed here.

4 Quantum Field Theory of Vortices

In the previous section we have seen that vortices in a two-dimensional superfluid at the absolute zero of temperature can be considered as point particles which are subject to the laws of quantum physics. It is therefore natural to ask whether vortices also admit a quantum-field-theoretic, or “second-quantized” description. In a recent letter, we have argued they indeed do and proposed the following quantum field theory [13]:

$$\mathcal{L} = \mathcal{L}_{\text{eff}} - ena_0 + \frac{e}{c} \mathbf{j} \cdot \mathbf{a} + \mathcal{L}_{\text{CS}}. \quad (77)$$

It consists of the nonrelativistic effective Lagrangian (24) describing the superfluid without vortices, *linearly* coupled via the particle number current $j_\mu = (n, \mathbf{j})$ to a vector field $a_\mu = (a_0, \mathbf{a})$ describing the vortices. This field is governed solely by a Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{1}{2c} \mathbf{a} \times \partial_0 \mathbf{a} - a_0 \nabla \times \mathbf{a}, \quad (78)$$

with the charge e determined by [13]

$$e^2 = \hbar c. \quad (79)$$

Our convention is such that, in contrast to j_μ , the dimensions of the time and space components of the vector field a_μ are the same. It is important to note that the coupling of the sound mode to the Chern-Simons field is linear and not minimal as in gauge theories. The coupled theory (77) therefore does not possess a gauge invariance involving a simultaneous local gauge transformation of the Chern-Simons field and the matter field. To appreciate why a minimal coupling is not feasible here, note that this would

inevitably result in the disappearance of the gapless sound mode because of the Higgs mechanism. This would disagree with the physics we wish to describe, namely the dynamics of vortices in a *compressible* superfluid film. For the gapless sound mode to survive a coupling to a gauge field, the coupling cannot be minimal so that there is no local gauge invariance in the full theory.

The appearance of a Chern-Simons term in (77) is expected for two reasons. First, as we discussed in the previous section, vortices in a superfluid at the absolute zero of temperature always move with the fluid. In other words, they have no independent dynamics. This is precisely also a basic property of a Chern-Simons term. It comes about because the zeroth component a_0 of the vector field is a Lagrange multiplier, demanding the equality

$$\nabla \times \mathbf{a} = -en, \quad (80)$$

or when integrated

$$\Phi = -eN, \quad (81)$$

where $N = \int d^2x n$ is the particle number and $\Phi = \int d^2x \nabla \times \mathbf{a}$ the “magnetic” flux associated with the Chern-Simons field. In the Coulomb gauge $\nabla \cdot \mathbf{a} = 0$, Eq. (80) can be easily solved to yield

$$\mathbf{a}(x) = -e\nabla \times \int d^2y G(\mathbf{x} - \mathbf{y}) n(t, \mathbf{y}), \quad (82)$$

where the Green function $G(\mathbf{x})$ is given in (45). The solution shows that the Chern-Simons field is entirely determined by the particle number density n and that it has, therefore, no independent dynamics.

The second reason why a Chern-Simons term is expected to appear is that it encodes the geometrical phase acquired by a vortex when it winds around a boson. The quantum-mechanical analog of such a term, representing the linking number [38] of a closed boson and vortex trajectory, was introduced in the problem by Arovas and Freire [11]. To calculate the geometrical phase $\gamma(\Gamma)$ we again evaluate the Wilson loop $W(\Gamma) = \exp[i\gamma(\Gamma)]$ obtained by integrating the Chern-Simons field \mathbf{a} around a closed path Γ in the superfluid,

$$\gamma(\Gamma) = \frac{e}{\hbar c} \oint_{\Gamma} d\mathbf{l} \cdot \mathbf{a}(x) = \frac{e^2}{\hbar c} \int d^2y \oint_{\Gamma} d\mathbf{l} \cdot [\nabla_{\mathbf{x}} \times \ln(|\mathbf{x} - \mathbf{y}|) n(t, \mathbf{y})], \quad (83)$$

where we substituted the explicit form (82) for \mathbf{a} . This geometrical phase is precisely the one in (65) obtained by Haldane and Wu [15]. Whereas in their quantum-mechanical description, the vortex being taken around the closed path Γ sees the encircled bosons as sources of geometric phase, here this counting is provided by the flux imparted to a particle by the Chern-Simons term [see Eq. (81)].

We next introduce external point vortices into the theory to determine their action and compare it with the action (48) obtained from the quantum-mechanical description of vortices. As we did there, we approximate the superfluid by an incompressible fluid and consider only the following terms of the Lagrangian (77):

$$\mathcal{L}_{\text{ext}} = -\bar{n} \left[\hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right] + \frac{e}{c} \mathbf{j} \cdot \mathbf{a}^{\text{ext}}. \quad (84)$$

The particle number current \mathbf{j} reads $\mathbf{j} = \bar{n}\mathbf{v}$ in this approximation, and the Chern-Simons field \mathbf{a}^{ext} is given by (82) with n replaced by the external vortex density

$$n_{\text{ext}}(x) = \sum_{\alpha} w_{\alpha} \delta[\mathbf{x} - \mathbf{X}^{\alpha}(t)]. \quad (85)$$

Explicitly,

$$\mathbf{a}^{\text{ext}}(x) = -\frac{e}{2\pi} \sum_{\alpha} w_{\alpha} \nabla \arctan \left(\frac{x_2 - X_2^{\alpha}(t)}{x_1 - X_1^{\alpha}(t)} \right), \quad (86)$$

or

$$a_i^{\text{ext}}(x) = \frac{e}{2\pi} \epsilon_{ij} \sum_{\alpha} w_{\alpha} \frac{x_j - X_j^{\alpha}(t)}{|\mathbf{x} - \mathbf{X}^{\alpha}(t)|^2}, \quad (87)$$

where $\mathbf{X}^{\alpha}(t)$ denotes the center of the α th vortex with winding number w_{α} . Note that the gauge $\nabla \cdot \mathbf{a} = 0$ is satisfied by the solution (87) only for $\mathbf{x} \neq \mathbf{X}^{\alpha}$, i.e., outside the vortex cores. The field equation for φ derived from (84) can be easily solved to yield

$$\varphi(x) = -\frac{e}{\hbar c} \int d^2y G(\mathbf{x} - \mathbf{y}) \nabla \cdot \mathbf{a}^{\text{ext}}(t, \mathbf{y}). \quad (88)$$

Upon taking the gradient of this equation, we obtain as velocity field $\mathbf{v} = (\hbar/m)\nabla\varphi$ the required form (46). Apart from a prefactor this expression coincides with the one for \mathbf{a}^{ext} in (87). This is because \mathbf{a}^{ext} can be written as a gradient of a scalar function [see (86)]. When substituting the solution (88) back into the Lagrangian (84), which is tantamount to integrating out the field φ , we recover precisely the action (48). We thus see that the theory (84) with its linear coupling to a Chern-Simons field correctly reproduces the two-dimensional action of point vortices in an incompressible superfluid.

Let us continue by considering the lowest order elastic scattering amplitude of two phonons calculated from the effective theory. In the frame where the sum of the momenta of the two incoming phonons is zero, the class of diagrams involving the exchange of Chern-Simons quanta represents the scattering of a phonon from a vortex. An analogous situation arises in the context of Aharonov-Bohm scattering, i.e., elastic scattering of a charged nonrelativistic particle from an infinitely thin magnetic flux tube. It was pointed out by Bergman and Lozano [39] that such a scattering process can also be described by a quantum field theory consisting of a nonrelativistic $|\psi|^4$ -theory (with zero chemical potential) coupled to a Chern-Simons term—albeit minimally.

At small energies and momenta, we can neglect the higher-order terms in the Lagrangian (77) and restrict ourselves to terms at most quadratic in the field φ :

$$\begin{aligned} \mathcal{L}^{(2)} = \frac{\bar{n}}{mc^2} & \left\{ \frac{1}{2} (\hbar \partial_0 \varphi)^2 - \frac{c^2}{2} (\hbar \nabla \varphi)^2 + e a_0 \left[\hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right] \right. \\ & \left. - \frac{e \hbar^2}{mc} \partial_0 \varphi \nabla \varphi \cdot \mathbf{a} \right\} + \mathcal{L}_{\text{CS}}. \end{aligned} \quad (89)$$

If we again introduce external vortices by replacing the Chern-Simons field \mathbf{a} with (87), the field equation for $\varphi(t, \mathbf{x} \neq \mathbf{X}^{\alpha})$ becomes

$$\partial_0^2 \varphi - c^2 \nabla^2 \varphi = 2 \frac{e}{mc} \partial_0 \nabla \varphi \cdot \mathbf{a}^{\text{ext}}, \quad (90)$$

where, in accordance with the derivation of the hydrodynamic equation (74), we omitted a term proportional to $\partial_0 \mathbf{a}^{\text{ext}}$. Given the form (87) of \mathbf{a}^{ext} , we recognize in the field equation (90) derived from our theory, the hydrodynamic equation (74) with the contribution from the vortex motion ignored. Since Eq. (74) was used as starting point for the quantum-mechanical description of the elastic scattering of a sound wave from a vortex, we expect the effective theory (89) to be the appropriate starting point for the quantum-field-theoretic description.

The propagators and vertices needed to calculate the scattering amplitude are readily obtained from that Lagrangian (89). In the gauge $\nabla \cdot \mathbf{a} = 0$, the nonzero components of the Chern-Simons propagator follow as

$$\begin{array}{c} 0 \quad k \quad i \\ \text{~~~~~} \text{~~~~~} \text{~~~~~} \end{array} : i\hbar G_{i0}(\mathbf{k}) = -i\hbar G_{0i}(\mathbf{k}) = -\hbar \epsilon_{ij} \frac{k_j}{\mathbf{k}^2}, \quad (91)$$

where \mathbf{k} is the wave vector. Note that there is no frequency dependence here. The phonon propagator reads

$$\begin{array}{c} p \\ \text{-----} \end{array} : i\hbar G(p) = \frac{mc^2}{\bar{n}} \frac{i\hbar}{p_0^2 - c^2 \mathbf{p}^2 + i\eta}, \quad (92)$$

where η is a small positive constant that has to be taken to zero after the integration over the loop energy p_0 has been carried out. In (92), \mathbf{p} denotes the momentum. As vertices we read off

$$\begin{array}{c} i \\ \text{~~~~~} \\ p \quad q \\ \text{-----} \end{array} : \frac{i}{\hbar} \Gamma_i(p, q) = -\frac{i}{\hbar} \frac{e\bar{n}}{m^2 c^3} (p_0 q_i + p_i q_0) \quad (93)$$

$$\begin{array}{c} 0 \\ \text{~~~~~} \\ p \quad q \\ \text{-----} \end{array} : \frac{i}{\hbar} \Gamma_0(p, q) = -\frac{i}{\hbar} \frac{e\bar{n}}{m^2 c^2} \mathbf{p} \cdot \mathbf{q} \quad (94)$$

$$\begin{array}{c} p \quad 0 \\ \text{-----} \end{array} : \frac{1}{\hbar} \frac{e\bar{n}}{mc^2} p_0. \quad (95)$$

In [13], we calculated the scattering amplitude $A(\theta, |\mathbf{p}|)$ to lowest order, with the result

$$\begin{array}{c} p \quad q \\ \text{-----} \\ \text{~~~~~} \\ \text{-----} \\ \tilde{p} \quad \tilde{q} \end{array} : iA^{(0)}(\theta, |\mathbf{p}|) = -\frac{1}{2} \frac{e^2 |\mathbf{p}|}{m^2 c^2} \frac{\sin \theta \cos \theta}{1 - \cos \theta}, \quad (96)$$

where, as in (76), θ is the scattering angle, and $|\mathbf{p}|$ the momentum of the incoming and scattered phonons. In deriving this, use is made of the following relations between the energy-momenta of the incoming and outgoing phonons

$$p_0 = \tilde{p}_0 = q_0 = \tilde{q}_0, \quad |\mathbf{p}| = |\tilde{\mathbf{p}}| = |\mathbf{q}| = |\tilde{\mathbf{q}}| = p_0/c, \quad \tilde{\mathbf{p}} = -\mathbf{p}, \quad \tilde{\mathbf{q}} = -\mathbf{q}, \quad (97)$$

and

$$\mathbf{p} \cdot \mathbf{q} = p^2 \cos(\theta), \quad (98)$$

in the frame defined by the condition that the sum of the momenta of the incoming phonons be zero. The normalization ζ of the external lines was determined in [13] to be given by the dimensionless factor

$$\zeta = \sqrt{\frac{mc^2}{2p_0 \bar{n} A}}. \quad (99)$$

where A is the surface area of the system. Apart from a kinematic factor, which is due to a different definition of the scattering amplitude in quantum mechanics and in quantum field theory, Eq. (96) agrees with the result (76) of Pitaevskii [16]. Hence, using the effective theory (89), we have reproduced the Born approximation of the scattering of a sound wave from a vortex.

For identical particles, one must add to the diagram in (96) the crossed diagram with the two outgoing lines exchanged. This yields the same result as in (96) with $\theta \rightarrow \theta - \pi$. Adding the two contributions, we find

$$A_{\text{tot}}^{(0)}(\theta, |\mathbf{p}|) = i \frac{\hbar \alpha}{m} \cot(\theta), \quad (100)$$

where we introduced the dimensionless parameter

$$\alpha = \xi |\mathbf{k}|, \quad (101)$$

which according to (35) was assumed to be small. Remember that $\mathbf{p} = \hbar \mathbf{k}$. The ratio \hbar/m in (100) is the circulation quantum. As observed by Sonin [37], the result (100) is identical to the lowest-order Aharonov-Bohm scattering amplitude, where the parameter α denotes the magnetic flux through the flux tube measured in units of Φ_0 , and m the mass of the scattered particle. The ratio \hbar/m should now be interpreted not as the circulation quantum, but as the dispersion constant [40] which determines the dispersion of a matter wave with wave vector \mathbf{k} describing a nonrelativistic particle of mass m ,

$$\omega(\mathbf{k}) = \frac{\hbar}{2m} \mathbf{k}^2. \quad (102)$$

In the quantum-field-theoretic approach to Aharonov-Bohm scattering [39], the contribution (100), which diverges for both $\theta = 0$ and π , is due to the exchange of a single Chern-Simons quantum. Since the Aharonov-Bohm scattering amplitude is known exactly, the next term in a perturbative calculation of this amplitude is known to be of order α^3 .

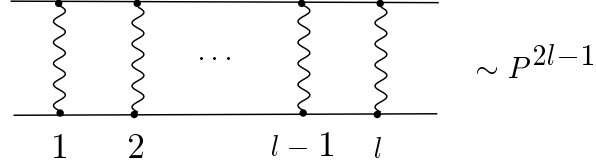


Figure 3: Contribution to the elastic scattering amplitude owing to the exchange of l Chern-Simons quanta. The incoming and outgoing phonons have an energy and momentum of the order cP and P , respectively.

5 One-Loop Corrections

In this section, we consider one-loop corrections to the tree amplitude (100). We are interested in the next order in an expansion in powers of the dimensionless quantity α introduced in (101). Because of the analogy with Aharonov-Bohm scattering, we expect this contribution to be of order α^3 and not of order α^2 . We calculate these contributions again using the effective Lagrangian (89).

To identify the relevant Feynman graphs, we apply—as is commonly done in the context of effective field theories—dimensional analysis to the higher-order contributions to the scattering amplitude. Following Weinberg [41], we assume that the energies and momenta of the two incoming and the two outgoing phonons are small and all of the same order, $p_0 = cP, |\mathbf{p}| = P$ say. As argued in that reference, the loop integrals will be dominated by contributions of the order P , so that one can make a perturbation in powers of P . More specifically, each energy and momentum appearing in the interaction vertices contributes a factor P . An internal phonon line therefore contributes a factor $1/P^2$ [see Eq. (92)], while an internal Chern-Simons line contributes a factor $1/P$ [see Eq. (91)]. An integration over a loop frequency and loop wave vectors contributes a factor P^3 in two space dimensions.

It is now easily shown that the exchange of l Chern-Simons quanta yields a contribution of the order of P^{2l-1} to the scattering amplitude (see Fig. 3), where we also took into account the energy-dependence contained in the normalization factor (99) of the external phonon lines. For the exchange of a single Chern-Simons quantum ($l = 1$), we recover the momentum dependence given in (100). Since the two vertices involved are both proportional to the charge e with which the phonons couple to the Chern-Simons field describing the vortices, these diagrams are proportional to e^{2l} . Including also the other constants contained in the vertices and the propagators, we find that the dimensionless expansion parameter for this type of diagrams is α^2 . These diagrams represent the scattering of a phonon from a vortex, and are the contributions we wish to evaluate.

For comparison, let us consider the case where instead of Chern-Simons quanta, phonons are exchanged. The interaction vertices involved in these contributions arise from the nonlinear terms in the effective theory which we have ignored in (89). Using similar arguments as the ones above, one can readily show that the exchange of l'

phonons yields a contribution to the phonon-phonon scattering amplitude of the order of $P^{3l'-1}$. To be more specific, the dimensionless expansion parameter is here

$$\beta = \frac{\xi}{\bar{n}} |\mathbf{k}|^3. \quad (103)$$

It is gratifying to see that the contributions which do not involve the Chern-Simons field describing the vortices have a different P -dependence.

From the above dimensional analysis, it follows that to order α^3 , the contributions to the scattering amplitude are of the form

$$A_{\text{tot}}^{(1)}(\theta, |\mathbf{p}|) = i \frac{h\alpha}{m} [C_0(\theta) + C_1(\theta, |\mathbf{p}|)\alpha^2], \quad (104)$$

where naively the coefficient of the second term, C_1 , is expected to be a function of the scattering angle θ only, in the same way as the coefficient of the first term,

$$C_0(\theta) = \cot(\theta), \quad (105)$$

depends only on θ . However, a one-loop diagram produces a nonanalytic term of the form $(h\alpha^3/m) \ln(|\mathbf{p}|)$, so that C_1 depends also on $|\mathbf{p}|$. Whether such logarithmic terms will appear can *a priori* not be determined solely by dimensional arguments because that function is dimensionless.

The calculation of the relevant one-loop corrections to the elastic phonon-phonon scattering amplitude is somewhat technical and therefore relegated to the Appendix. The final result for the coefficient $C_1(\theta, |\mathbf{p}|)$ introduced in (104), which we split as

$$C_1(\theta, |\mathbf{p}|) := C_1(\theta) + C_{\text{na}}(|\mathbf{p}|) \quad (106)$$

reads

$$\begin{aligned} C_1(\theta) = & -\frac{1}{8\pi^2} \left\{ \frac{5}{3} + \sin(\tfrac{1}{2}\theta) \cos(\tfrac{1}{2}\theta) [\tfrac{1}{2}\pi - 2 \sin(\tfrac{1}{2}\theta) \cos(\tfrac{1}{2}\theta)] \right. \\ & + \sin^3(\tfrac{1}{2}\theta) \left[\ln \left(\frac{1 - \sin(\tfrac{1}{2}\theta)}{\cos(\tfrac{1}{2}\theta)} \right) - \frac{1}{2} \left(\pi + \frac{\theta - \pi}{\cos(\tfrac{1}{2}\theta)} \right) \right] \\ & \left. + \cos^3(\tfrac{1}{2}\theta) \left[\ln \left(\frac{\sin(\tfrac{1}{2}\theta)}{1 + \cos(\tfrac{1}{2}\theta)} \right) - \frac{1}{2} \left(\pi - \frac{\theta}{\sin(\tfrac{1}{2}\theta)} \right) \right] \right\}, \end{aligned} \quad (107)$$

and

$$C_{\text{na}}(|\mathbf{p}|) = -\frac{1}{16\pi^3} \int_0^1 dy \frac{1}{y^2 - 1}. \quad (108)$$

This last contribution, which is obtained from Eq. (A.25) with the crossed term included, is logarithmically diverging. It is independent of the scattering angle. A closer inspection reveals that the divergence is an infrared divergence arising when the momentum of one of the Chern-Simons propagators in the diagram depicted in Fig. 7

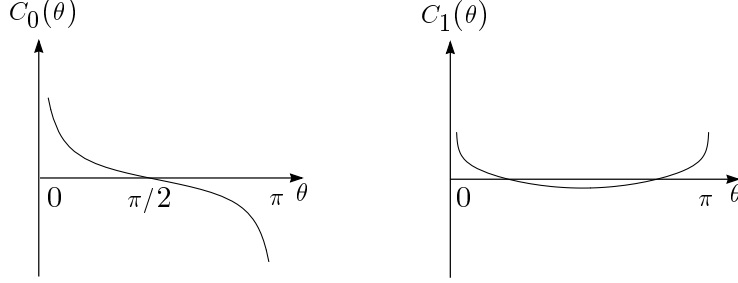


Figure 4: Graphical representation of the $\mathcal{O}(\alpha)$ -contribution $C_0(\theta)$ and the $\mathcal{O}(\alpha^3)$ -contribution $C_1(\theta)$ to the elastic phonon-phonon scattering amplitude.

tends to zero. To regularize the integral, we give the Chern-Simons field a small mass μ ; $C_{\text{na}}(|\mathbf{p}|)$ then becomes

$$\begin{aligned}
C_{\text{na}}(|\mathbf{p}|) &= -\frac{1}{32\pi^3} \int_0^1 dy \frac{y-1}{(y+1)(y-\sqrt{1-\mu^2 c^2/|\mathbf{p}^2|})^2} \\
&\quad + \frac{1}{32\pi^3} \int_1^\infty dy \frac{y-1}{(y+1)(y-\sqrt{1-\mu^2 c^2/|\mathbf{p}^2|})^2} \\
&\rightarrow \frac{1}{16\pi^3} \left[\ln \left(\frac{2|\mathbf{p}|}{\mu c} \right) - \frac{1}{2} \right].
\end{aligned} \tag{109}$$

This introduces an arbitrary scale factor into the problem. Frequently, such a scale factor arises instead when ultraviolet divergences are present. Logarithmic divergences of this kind have to be eliminated by a redefinition of certain parameters of the effective theory at the expense of the appearance of an arbitrary renormalization scale [41]. The nonrelativistic effective theory considered here is ultraviolet finite to this order, and needs, therefore, not to be renormalized. The arbitrary scale factor μ arises here from the infrared region.

A graphical representation of the analytic part of the one-loop result, $C_1(\theta)$, together with the tree result is given in Fig. 4. It shows that whereas the tree contribution $C_0(\theta)$ is antisymmetric about $\theta = \pi/2$, the $\mathcal{O}(\alpha^3)$ -contributions are symmetric. Both are seen to diverge for $\theta = 0$ and π .

6 Discussion

We have extended the analysis of the quantum field theory recently proposed by these authors [13] to describe vortices in a superfluid film at the absolute zero of temperature.

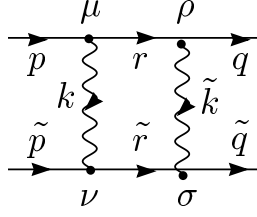


Figure 5: One-loop diagram I contributing to the scattering amplitude.

The theory consists of the effective action of phonons coupled to a Chern-Simons term. Two of its salient features are that, first, the effective action describing the phonons is invariant under Galilei transformations—as is required for a nonrelativistic system—and, second, the coupling to the Chern-Simons terms is linear and not minimal. We have demonstrated that various known facts about two-dimensional quantum vortices are correctly reproduced by the quantum field theory.

We have further shown how it can be used to calculate one-loop contributions to the amplitude for the elastic scattering of phonons from a vortex. We have applied dimensional analysis of the type commonly used in the context of effective field theories to identify the relevant Feynman graphs and have calculated these. We have shown that the one-loop corrections are ultraviolet finite, and that an arbitrary scale factor was introduced by an infrared divergence.

By minimally coupling the Goldstone field φ to electrodynamics, the theory discussed here may be extended to describe vortices in a superconducting film.

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Appendix

In this Appendix, we calculate the contributions to the elastic phonon-phonon scattering amplitude proportional to α^3 . These contributions arise from the three one-loop diagrams depicted in Figs. 5-7. Since the dimensional analysis of the previous section already fixed the dimensionful prefactors appearing in the scattering amplitude, we can simplify matters by setting all dimensionful parameters to one,

$$\hbar = c = m = \bar{n} = 1. \quad (\text{A.1})$$

The Feynman rules (91)-(94) applied to the diagram depicted in Fig. 5, which we

refer to as I, yield

$$I = \zeta^4 \int \frac{d^3 r}{(2\pi)^3} G(r) G(\tilde{r}) G_{\mu\nu}(k) G_{\rho\sigma}(\tilde{k}) \Gamma_\mu(p, -r) \Gamma_\nu(\tilde{p}, -\tilde{r}) \Gamma_\rho(r, -q) \Gamma_\sigma(\tilde{r}, -\tilde{q}), \quad (\text{A.2})$$

where because of conservation of energy and momentum at the vertices we have in addition to (97) the following relations

$$\tilde{r}_0 = 2p_0 - r_0, \quad k_0 = p_0 - r_0, \quad \tilde{k}_0 = r_0 - p_0, \quad \tilde{\mathbf{r}} = -\mathbf{r}, \quad \mathbf{k} = \mathbf{p} - \mathbf{r}, \quad \tilde{\mathbf{k}} = \mathbf{r} - \mathbf{q}. \quad (\text{A.3})$$

The summation over the indices in (A.2) (using an Euclidean metric) gives

$$\begin{aligned} G_{\mu\nu}(k) G_{\rho\sigma}(\tilde{k}) \Gamma_\mu(p, -r) \Gamma_\nu(\tilde{p}, -\tilde{r}) \Gamma_\rho(r, -q) \Gamma_\sigma(\tilde{r}, -\tilde{q}) = \\ -\frac{1}{\mathbf{k}^2 \tilde{\mathbf{k}}^2} [\mathbf{p} \cdot \mathbf{r} (p_0 \tilde{\mathbf{r}} \times \mathbf{k} - \tilde{r}_0 \mathbf{p} \times \mathbf{k}) + \mathbf{p} \cdot \tilde{\mathbf{r}} (p_0 \mathbf{r} \times \mathbf{k} + r_0 \mathbf{p} \times \mathbf{k})] \\ \times [\mathbf{q} \cdot \mathbf{r} (p_0 \tilde{\mathbf{r}} \times \tilde{\mathbf{k}} - \tilde{r}_0 \mathbf{q} \times \tilde{\mathbf{k}}) + \mathbf{q} \cdot \tilde{\mathbf{r}} (p_0 \mathbf{r} \times \tilde{\mathbf{k}} + r_0 \mathbf{q} \times \tilde{\mathbf{k}})]. \end{aligned} \quad (\text{A.4})$$

Substituting this in (A.2) and using the relations (A.3), we obtain the expression

$$I = 4 \int \frac{d^3 r}{(2\pi)^3} \frac{1}{(r_0^2 - \mathbf{r}^2 + i\eta)[(2p_0 - r_0)^2 - \mathbf{r}^2 + i\eta]} \frac{\mathbf{p} \cdot \mathbf{r} \mathbf{p} \times \mathbf{r} \mathbf{q} \cdot \mathbf{r} \mathbf{q} \times \mathbf{r}}{(\mathbf{p} - \mathbf{r})^2 (\mathbf{q} - \mathbf{r})^2}. \quad (\text{A.5})$$

With the help of contour integration, the integral over the loop energy r_0 is readily carried out with the result

$$I = i \int \frac{d^2 r}{(2\pi)^2} \frac{\mathbf{p} \cdot \mathbf{r} \mathbf{p} \times \mathbf{r} \mathbf{q} \cdot \mathbf{r} \mathbf{q} \times \mathbf{r}}{|\mathbf{r}| (\mathbf{r}^2 - p_0^2) (\mathbf{p} - \mathbf{r})^2 (\mathbf{q} - \mathbf{r})^2}. \quad (\text{A.6})$$

The integral over the angle α is most easily carried out by introducing the variable $z = \exp(i\alpha)$ and performing a contour integration along the unit circle. The poles one encounters are located at

$$z = 0, \quad z = y, \quad z = 1/y, \quad z = e^{i\theta} y, \quad z = e^{i\theta}/y, \quad (\text{A.7})$$

where y stands for

$$y = \frac{|\mathbf{r}|}{p_0}. \quad (\text{A.8})$$

Since only those poles lying inside the unit circle are to be included, the regions $0 \leq y < 1$ and $1 < y < \infty$ have to be treated separately. Adding the two contributions, we find

$$I = \frac{ip_0^3}{8} \int_0^1 \frac{dy}{2\pi} (y^2 + 1) \frac{\cos(2\theta) + 2 \cos(2\theta) y^2 - [1 + 2 \cos(\theta)] y^4}{1 - 2 \cos(\theta) y^2 + y^4}. \quad (\text{A.9})$$

The remaining integral over y is elementary and yields

$$I = \frac{-ip_0^3}{16\pi} \left\{ \frac{10}{3} + \frac{14}{3} \cos(\theta) + 4 \cos^3(\frac{1}{2}\theta) \ln[\tan(\frac{1}{4}\theta)] \right\}. \quad (\text{A.10})$$

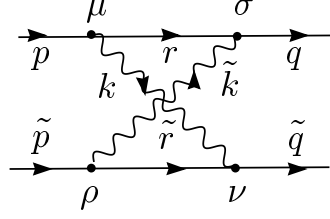


Figure 6: One-loop diagram II contributing to the scattering amplitude.

This result is valid for $0 \leq \theta \leq 2\pi$.

We continue with the evaluation of the second one-loop diagram depicted in Fig. 6. For this diagram II we find

$$\Pi = \zeta^4 \int \frac{d^3 r}{(2\pi)^3} G(r) G(\tilde{r}) G_{\mu\nu}(k) G_{\rho\sigma}(\tilde{k}) \Gamma_\mu(p, -r) \Gamma_\nu(\tilde{r}, -\tilde{q}) \Gamma_\rho(\tilde{p}, -\tilde{r}) \Gamma_\sigma(r, -q), \quad (\text{A.11})$$

where now

$$\tilde{r}_0 = r_0, \quad k_0 = p_0 - r_0, \quad \tilde{k}_0 = p_0 - r_0, \quad \tilde{\mathbf{r}} = \mathbf{r} - \mathbf{p} - \mathbf{q}, \quad \mathbf{k} = \mathbf{p} - \mathbf{r}, \quad \tilde{\mathbf{k}} = \mathbf{q} - \mathbf{r}. \quad (\text{A.12})$$

Explicitly,

$$\begin{aligned} G_{\mu\nu}(k) G_{\rho\sigma}(\tilde{k}) \Gamma_\mu(p, -r) \Gamma_\nu(\tilde{r}, -\tilde{q}) \Gamma_\rho(\tilde{p}, -\tilde{r}) \Gamma_\sigma(r, -q) = \\ \frac{1}{\mathbf{k}^2 \tilde{\mathbf{k}}^2} [\mathbf{p} \cdot \mathbf{r} (p_0 \tilde{\mathbf{r}} \times \mathbf{k} - \tilde{r}_0 \mathbf{q} \times \mathbf{k}) + \mathbf{q} \cdot \tilde{\mathbf{r}} (p_0 \mathbf{r} \times \mathbf{k} + r_0 \mathbf{p} \times \mathbf{k})] \\ \times [\mathbf{p} \cdot \tilde{\mathbf{r}} (p_0 \mathbf{r} \times \tilde{\mathbf{k}} + r_0 \mathbf{q} \times \tilde{\mathbf{k}}) + \mathbf{q} \cdot \mathbf{r} (p_0 \tilde{\mathbf{r}} \times \tilde{\mathbf{k}} - \tilde{r}_0 \mathbf{p} \times \tilde{\mathbf{k}})]. \end{aligned} \quad (\text{A.13})$$

Substituting this in (A.11) and using the relations (A.12), we arrive at

$$\begin{aligned} \Pi = & -\frac{1}{4p_0^2} \int \frac{d^3 r}{(2\pi)^3} \frac{(p_0 + r_0)^2}{(r_0^2 - \mathbf{r}^2 + i\eta)[r_0^2 - (\mathbf{r} - \mathbf{p} - \mathbf{q})^2 + i\eta]} \frac{1}{(\mathbf{p} - \mathbf{r})^2 (\mathbf{q} - \mathbf{r})^2} \\ & \times [\mathbf{p} \cdot \mathbf{r} (\mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r}) - \mathbf{q} \cdot (\mathbf{r} - \mathbf{p} - \mathbf{q}) \mathbf{p} \times \mathbf{r}] \\ & \times [-\mathbf{q} \cdot \mathbf{r} (-\mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{r}) + \mathbf{p} \cdot (\mathbf{r} - \mathbf{p} - \mathbf{q}) \mathbf{q} \times \mathbf{r}]. \end{aligned} \quad (\text{A.14})$$

Let us first carry out the integral over the loop energy r_0 :

$$\int \frac{dr_0}{2\pi} \frac{(p_0 + r_0)^2}{(r_0^2 - \mathbf{r}^2 + i\eta)(r_0^2 - \mathbf{v}^2 + i\eta)} = -\frac{i}{2} \frac{(p_0 - \mathbf{r})^2}{|\mathbf{r}|(\mathbf{r}^2 - \mathbf{v}^2)} - \frac{i}{2} \frac{(p_0 - |\mathbf{v}|)^2}{|\mathbf{v}|(\mathbf{v}^2 - \mathbf{r}^2)}, \quad (\text{A.15})$$

where we introduced the abbreviation $\mathbf{v} = \mathbf{r} - \mathbf{p} - \mathbf{q}$. The first term in (A.15) gives a contribution Π_1 to Π and the second term a contribution Π_2 , such that $\Pi = \Pi_1 + \Pi_2$. It turns out that both contributions are identical, so that $\Pi = 2\Pi_1$. We perform the angle integration again by introducing the variable $z = \exp(i\alpha)$ and carrying out a

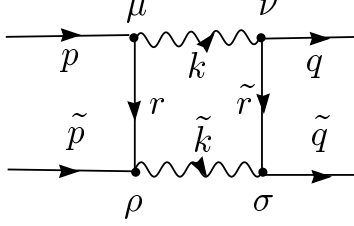


Figure 7: One-loop diagram III contributing to the scattering amplitude.

contour integration along the unit circle. Besides poles located at $z = y, e^{i\theta}y$ and $z = 1/y, e^{i\theta}/y$, with y defined in (A.8), there are also two poles at

$$z_{1,2} = \frac{1 + e^{i\theta} \pm \sqrt{(e^{i\theta} + 1)^2 - 4e^{i\theta}y^2}}{2y}. \quad (\text{A.16})$$

The contributions of the first two pairs cancel, so that we only have to consider the contributions stemming from the poles z_1 and z_2 . For $y \geq \cos(\frac{1}{2}\theta)$ these poles lie precisely on the unit circle, $|z_{1,2}| = 1$ and it is *a priori* not clear how to proceed. A closer (numerical) inspection reveals that the contributions from z_1 and z_2 cancel there. This leaves us with region $y < \cos(\frac{1}{2}\theta)$, where $|z_1| < 1$ and $|z_2| > 1$. In this way, we obtain with $t = \cos(\frac{1}{2}\theta)$ the result

$$\begin{aligned} \Pi &= \frac{-ip_0^3}{16\pi} \int_0^t dy \frac{4t(1-y)[y^2(t^2-1) + t^2]}{(1+y)\sqrt{t^2-y^2}} \\ &= \frac{-ip_0^3}{16\pi} \left\{ -4 \sin^2(\tfrac{1}{2}\theta) \cos^2(\tfrac{1}{2}\theta) + \frac{\pi}{2} \cos(\tfrac{1}{2}\theta) [4 - 5 \cos^2(\tfrac{1}{2}\theta) - \cos^4(\tfrac{1}{2}\theta)] \right. \\ &\quad \left. + 2\theta \cot(\tfrac{1}{2}\theta) \cos(\theta) \right\}, \end{aligned} \quad (\text{A.17})$$

which is valid for $-\pi \leq \theta \leq \pi$.

The last one-loop contribution to the phonon-phonon scattering amplitude is depicted in Fig. 7. It stands for

$$\text{III} = \zeta^4 \int \frac{d^3r}{(2\pi)^3} G(r)G(\tilde{r})G_{\mu\nu}(k)G_{\rho\sigma}(\tilde{k})\Gamma_\mu(p, -r)\Gamma_\nu(-q, -\tilde{r})\Gamma_\rho(\tilde{p}, r)\Gamma_\sigma(-\tilde{q}, \tilde{r}), \quad (\text{A.18})$$

where now we have the relations

$$\tilde{r}_0 = -r_0, k_0 = p_0 - r_0, \tilde{k}_0 = p_0 + r_0, \tilde{\mathbf{r}} = \mathbf{p} - \mathbf{q} - \mathbf{r}, \mathbf{k} = \mathbf{p} - \mathbf{r}, \tilde{\mathbf{k}} = \mathbf{r} - \mathbf{p}. \quad (\text{A.19})$$

More explicitly,

$$\begin{aligned} G_{\mu\nu}(k)G_{\rho\sigma}(\tilde{k})\Gamma_\mu(p, -r)\Gamma_\nu(-q, -\tilde{r})\Gamma_\rho(\tilde{p}, r)\Gamma_\sigma(-\tilde{q}, \tilde{r}) = \\ \frac{1}{k^2\tilde{k}^2} [\mathbf{p} \cdot \mathbf{r}(p_0 \tilde{\mathbf{r}} \times \mathbf{k} + \tilde{r}_0 \mathbf{q} \times \mathbf{k}) - \mathbf{q} \cdot \tilde{\mathbf{r}}(p_0 \mathbf{r} \times \mathbf{k} + r_0 \mathbf{p} \times \mathbf{k})] \\ \times [\mathbf{p} \cdot \mathbf{r}(p_0 \tilde{\mathbf{r}} \times \tilde{\mathbf{k}} - \tilde{r}_0 \mathbf{q} \times \tilde{\mathbf{k}}) - \mathbf{q} \cdot \tilde{\mathbf{r}}(p_0 \mathbf{r} \times \tilde{\mathbf{k}} - r_0 \mathbf{p} \times \tilde{\mathbf{k}})]. \end{aligned} \quad (\text{A.20})$$

In this way, (A.18) becomes

$$\text{III} = \frac{-1}{4p_0^2} \int \frac{d^3 r}{(2\pi)^3} \frac{p_0^2 - r_0^2}{(r_0^2 - \mathbf{r}^2 + i\eta)[r_0^2 - (\mathbf{p} - \mathbf{q} - \mathbf{r})^2 + i\eta]} \times \frac{[\mathbf{p} \cdot \mathbf{r}(\mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r}) + \mathbf{q} \cdot (\mathbf{p} - \mathbf{q} - \mathbf{r})\mathbf{p} \times \mathbf{r}]^2}{(\mathbf{p} - \mathbf{r})^4}. \quad (\text{A.21})$$

We again first carry out the integral over the loop energy r_0 ,

$$\int \frac{dr_0}{2\pi} \frac{p_0^2 - r_0^2}{(r_0^2 - \mathbf{r}^2 + i\eta)(r_0^2 - \mathbf{w}^2 + i\eta)} = \frac{i}{2} \frac{\mathbf{r}^2 - p_0^2}{|\mathbf{r}|(\mathbf{r}^2 - \mathbf{w}^2)} + \frac{i}{2} \frac{\mathbf{w}^2 - p_0^2}{|\mathbf{w}|(\mathbf{w}^2 - \mathbf{r}^2)}, \quad (\text{A.22})$$

where $\mathbf{w} = \mathbf{r} + \mathbf{q} - \mathbf{p}$. The first (second) term gives a contribution $\text{III}_{1(2)}$, such that $\text{III} = \text{III}_1 + \text{III}_2$. The angle integral is again evaluated by a contour integration along the unit circle. With y as defined in (A.8), the integrand of III_1 has poles at $z = y, 1/y$ as well as at

$$z_{3,4} = \frac{1 - e^{i\theta} \mp \sqrt{(e^{i\theta} - 1)^2 + 4e^{i\theta}y^2}}{2y}, \quad (\text{A.23})$$

while the integrand of III_2 has poles at $z = e^{i\theta}y, e^{i\theta}/y$ and at

$$z_{5,6} = \frac{-1 + e^{i\theta} \mp \sqrt{(e^{i\theta} - 1)^2 + 4e^{i\theta}y^2}}{2y}. \quad (\text{A.24})$$

In III_2 , we made the change of integration variable $\mathbf{r} \rightarrow \mathbf{w}$. The contributions from the poles $z = y, 1/y, e^{i\theta}y, e^{i\theta}/y$ can be calculated in a straightforward manner. They cause a logarithmic-diverging integral

$$\text{III}_{\text{na}} = \frac{ip_0^3}{4\pi} \sin^2(\tfrac{1}{2}\theta) \int_0^1 dy \frac{1}{y^2 - 1}, \quad (\text{A.25})$$

which is discussed in the main text. The remaining poles z_3, z_4, z_5 , and z_6 which we have to consider lie precisely on the unit circle for $y \geq \sin(\tfrac{1}{2}\theta)$. A careful numerical analysis reveals that their contributions cancel here. This leaves us with the region $y \leq \sin(\tfrac{1}{2}\theta)$, where only the poles z_3 and z_6 have to be included since $|z_{3,6}| < 1$ while $|z_{4,5}| > 1$. Both z_3 and z_6 give the same real part, while their imaginary parts cancel. With s denoting $\sin(\tfrac{1}{2}\theta)$, we finally obtain for diagram III

$$\begin{aligned} \text{III} - \text{III}_{\text{na}} &= \\ \frac{ip_0^3}{16\pi} \int_0^s dy &\frac{4s(y^2 - 1)[-s^2 + y^2 + (1 - s^2)y^4]}{\sqrt{s^2 - y^2} \left[y^4 - 2sy^2 \left(s - \sqrt{s^2 - y^2} \right) + \left(s - \sqrt{s^2 - y^2} \right)^2 \right]^2} \\ &\times \left(8s^4 - 8s^2y^2 + y^4 - 8s^3\sqrt{s^2 - y^2} + 4sy^2\sqrt{s^2 - y^2} \right) = \\ \frac{-ip_0^3}{16\pi} &\left\{ 2\pi |\sin(\tfrac{1}{2}\theta)| \cos(\tfrac{1}{2}\theta) - \frac{\pi}{2} |\sin(\tfrac{1}{2}\theta)| \left[4 - \sin^2(\tfrac{1}{2}\theta) - \sin^4(\tfrac{1}{2}\theta) \right] \right\}, \end{aligned} \quad (\text{A.26})$$

where θ is to be restricted to the values $-\pi \leq \theta \leq \pi$.

Since we are dealing with identical particles, we have to include the crossed diagrams whose contributions are obtained from the uncrossed diagrams by replacing $\theta \rightarrow \theta - \pi \bmod(2\pi)$. More specifically, restricting ourselves to the values $0 \leq \theta \leq \pi$, we have to replace θ with $\theta + \pi$ in the result (A.9) of diagram I, and with $\theta - \pi$ in the results (A.17) and (A.26) of diagram II and III. The final expression for the one-loop contributions to the elastic phonon-phonon scattering proportional to α^3 is given in (107) and (109).

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